

Global regular axially symmetric solutions to the Navier-Stokes equations in a periodic cylinder

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Abstract. We examine the axially symmetric solutions to the Navier-Stokes equations in a periodic cylinder with the slip boundary conditions on the lateral part of the boundary. Having initial velocity $v(0) \in H^1(\Omega)$ we prove the existence of local regular solutions such that $v \in W_2^{2,1}(\Omega^{T_*})$, $\nabla p \in L_2(\Omega^{T_*})$, where T_* is so small that

$$(1) \quad cT_*^{1/2} \|v(0)\|_{H^1(\Omega)} \leq 1.$$

If $v \in W_2^{2,1}(\Omega^{T_*})$ then swirl $u = rv_\varphi$ belongs to $C^{1/2,1/4}(\Omega^{T_*})$ and vanishes on the axis of symmetry.

Let v_r, v_φ, v_z be the cylindrical components of velocity and $\chi = v_{r,z} - v_{z,r}$.

Introducing the quantity ($c \geq 1$)

$$(2) \quad \begin{aligned} \alpha &= c \left(\|v(0)\|_{H^1(\Omega)} + \left\| \frac{v_\varphi(0)}{\sqrt{r}} \right\|_{L_4(\Omega)} + \left\| \frac{v_\varphi(0)}{r} \right\|_{L_3(\Omega)} + \left\| \frac{\chi(0)}{r} \right\|_{L_2(\Omega)} \right) \\ &\geq \|v(0)\|_{H^1(\Omega)} \end{aligned}$$

we derive the estimate

$$(3) \quad \|v(T_*)\|_{H^1(\Omega)} \leq \|v\|_{V_2^1(\Omega^{T_*})} \leq \alpha.$$

Defining T by the relation

$$(4) \quad cT^{1/2}\alpha \leq 1$$

we have that $T < T_*$. Then (3) yields that

$$(5) \quad \|v(T)\|_{H^1(\Omega)} \leq \alpha.$$

Starting from time $t = T$ with initial data satisfying (5) and repeating the above considerations the existence in the interval $[T, 2T]$ is proved and also estimate (3) holds, so

$$(6) \quad \|v(2T)\|_{H^1(\Omega)} \leq \alpha.$$

Since estimate (3) is a global a priori estimate the extension can be performed step by step in all intervals $(kT, (k+1)T)$, $k \in \mathbb{N}$. Therefore, we prove the existence of such solutions that

$$v \in W_2^{2,1}(\Omega \times (kT, (k+1)T)), \quad \nabla p \in L_2(\Omega \times (kT, (k+1)T)), \quad k \in \mathbb{N} \cup \{0\},$$

where T is determined by (4) with α introduced in (2).

1. Introduction

We consider the axially symmetric solutions to the problem

$$(1.1) \quad \begin{aligned} v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v, p) &= 0 && \text{in } \Omega^T = \Omega \times (0, T), \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} &= 0 && \text{on } S^T = S \times (0, T), \\ \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ \text{periodic boundary conditions} &&& \text{on } S_2^T, \\ v|_{t=0} &= v_0, && \text{in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a cylinder with boundary $S = S_1 \cup S_2$, S_1 is parallel to the axis of the cylinder, but S_2 is perpendicular, $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x, t) \in \mathbb{R}$ the pressure, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates such that x_3 axis is the axis of the cylinder, \bar{n} is the unit outward normal vector to S_1 , $\bar{\tau}_\alpha$, $\alpha = 1, 2$, is the tangent vector to S_1 and the dot denotes the scalar product in \mathbb{R}^3 .

By $\mathbb{T}(v, p)$ we denote the stress tensor of the form

$$(1.2) \quad \mathbb{T}(v, p) = \nu \mathbb{D}(v) - pI,$$

where $\nu > 0$ is the constant viscosity coefficient, $\mathbb{D}(v)$ is the dilatation tensor of the form

$$(1.3) \quad \mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{ij=1,2}$$

and I is the unit matrix.

We are interested to examine axially symmetric solutions to problem (1.1). Hence we introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r \cos \varphi, x_2 = r \sin \varphi, x_3 = z$.

We assume that Ω is axially symmetric bounded cylinder with x_3 axis as the axis of symmetry. Let R and a be given positive numbers. Then

$$\Omega = \{x \in \mathbb{R}^3 : r < R, |z| < a\}.$$

Hence

$$S_1 = \{x \in \mathbb{R}^3 : r = R, |z| \leq a\}$$

and

$$S_2 = \{x \in \mathbb{R}^3 : r \leq R, z \text{ is equal either } -a \text{ or } a\}.$$

Let us introduce the vectors

$$\bar{e}_r = (\cos \varphi, \sin \varphi, 0), \quad \bar{e}_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad \bar{e}_z = (0, 0, 1),$$

which are connected with cylindrical coordinates. Then the cylindrical coordinates of velocity are defined by the relations

$$(1.4) \quad v_r = v \cdot \bar{e}_r, \quad v_\varphi = v \cdot \bar{e}_\varphi, \quad v_z = v \cdot \bar{e}_z.$$

Definition 1.1. By the axially symmetric solution to problem (1.1) we mean such solution that

$$(1.5) \quad v_{r,\varphi} = v_{\varphi,\varphi} = v_{z,\varphi} = p_{,\varphi} = 0.$$

The aim of this paper is to prove the existence of global regular axially symmetric solutions to problem (1.1) with large swirl u , which is defined by

$$(1.6) \quad u = rv_\varphi.$$

We have to recall that behavior of axially symmetric solutions is essentially different near and far from the axis of symmetry. An appropriate a priori estimate near the axis is found in [Z1]. In this paper we show

a corresponding estimate outside of the axis. Hence the estimate in the whole domain Ω follows. The estimate, we are looking for, is for the norm $\|v(t)\|_{H^1(\Omega)}$, $t \in \mathbb{R}_+$. The estimate is sufficient to prove the existence of global regular axially symmetric solutions to (1.1).

To formulate the main results of this paper we need

Assumptions:

1. Let $v_0 \in L_2(\Omega)$ and let d_1 be a positive constant such that $\|v_0\|_{L_2(\Omega)} \leq d_1$.
2. Let $u_0 \in L_\infty(\Omega)$ and let d_2 be a positive constant such that $\|u_0\|_{L_\infty(\Omega)} \leq d_2$.
3. Let $\zeta_1 = \zeta_1(r)$ be a smooth cut off function such that $\zeta_1(r) = 1$ for $r \leq r_0$ and $\zeta_1(r) = 0$ for $2r_0 \leq r < R$. Next $\zeta_2(r) = 1$ for $r \geq r_0$ and $\zeta_2(r) = 0$ for $r \leq \frac{r_0}{2}$. Finally $\zeta_3(r) = 1$ for $r \geq 2r_0$ and $\zeta_3(r) = 0$ for $r \leq r_0$. Moreover, $\{\zeta_1(r), \zeta_3(r)\}$ is a partition of unity in $[0, R]$. Let $\Omega_{\zeta_i} = \Omega \cap \text{supp } \zeta_i$ and $i = 1, 2, 3$.
4. Let $u_0 \in C^\alpha(\Omega)$, $\alpha \leq \frac{1}{2}$. Let r_0 be so small that

$$(1.7) \quad \|u_0\|_{C^\alpha(\Omega_{\zeta_1})} r_0^\alpha \leq \sqrt[4]{\frac{5}{2}}^\nu$$

5. Assume that

$$\left\| \frac{v_\varphi(0)}{\sqrt{r}} \right\|_{L_4(\Omega_{\zeta_1})} < \infty, \quad \left\| \frac{\chi(0)}{r} \right\|_{L_2(\Omega_{\zeta_1})} < \infty.$$

6. Let

$$\begin{aligned} A_1 &= \varphi(d_1, d_2, 1/r_0) \left(1 + \left\| \frac{v_\varphi(0)}{\sqrt{r}} \right\|_{L_4(\Omega_{\zeta_1})} + \left\| \frac{\chi(0)}{r} \right\|_{L_2(\Omega_{\zeta_1})} \right), \\ A_2 &= \varphi(d_1, 1/r_0) (1 + \|v_\varphi(0)\|_{L_3(\Omega_{\zeta_2})} + \|\chi(0)\|_{L_2(\Omega_{\zeta_2})}), \\ A &= A_1 + A_2, \\ A_3 &= \varphi(A_1)(d_1 + d_2) + c \left(\|v_\varphi(0)\|_{H^1(\Omega)} + \left\| \frac{v_\varphi(0)}{r} \right\|_{L_{\frac{27}{10}}(\Omega)} \right), \end{aligned}$$

where φ is an increasing positive function.

Theorem A. *Let assumptions 1–6 hold. Let $u_0 \in C^\alpha(\Omega)$, $\alpha \leq 1/2$, $v_0 \in H^1(\Omega)$. Let $\alpha = A_1 + A_2 + A_3$. Let c_* be some positive constant. Let T be such that $c_* T^{1/2} \alpha \leq 1$. Then there exists a global regular axially symmetric solution to (1.1) such that $v \in W_2^{2,1}(\Omega \times (kT, (k+1)T))$, $\nabla p \in L_2(\Omega \times (kT, (k+1)T))$ $k \in \mathbb{N} \cup \{0\}$ and*

$$(1.9) \quad \begin{aligned} &\|v\|_{W_2^{2,1}(\Omega \times (kT, (k+1)T))} + \|\nabla p\|_{L_2(\Omega \times (kT, (k+1)T))} \\ &\leq \varphi(\alpha, \|u_0\|_{C^{1/2}(\Omega)}) + c\|v_0\|_{H^1(\Omega)}. \end{aligned}$$

Theorem B. *In the above case we have uniqueness of solutions.*

Now we shall describe shortly the proof of Theorem A. Having that $v(0) \in H^1(\Omega)$ we prove a local existence of solutions to problem (1.1) such that $v \in W_2^{2,1}(\Omega^{T_*})$, $\nabla p \in L_2(\Omega^{T_*})$, where T_* is so small that

$$(1.8) \quad c_* T_*^{1/2} \|v(0)\|_{H^1(\Omega)} \leq 1,$$

where constant c_* is described in (3.3).

Then it follows that $v \in L_r(0, T_*; L_p(\Omega))$, where $\frac{3}{p} + \frac{2}{r} = \frac{1}{2}$. Next Theorem 5.4 from [Z4] implies that $u \in C(0, T_*; C^{1/2}(\Omega_{\zeta_1}))$, where Ω_{ζ_1} is described in the above Assumption 3.

Since u is continuous, Ω_{ζ_1} belongs to a cylinder with radius $2r_0$ and u vanishes on the axis of symmetry we obtain for r_0 sufficiently small that

$$(1.9) \quad \|u\|_{C(0, T_*; C^{1/2}(\Omega_{\zeta_1}))} \leq \sqrt[4]{\frac{5}{4}} \nu.$$

Then Lemma 6.4 from [Z1] yields the estimate

$$(1.10) \quad \|v'\|_{V_2^1(\Omega_{\zeta_1}^{T_*})} \leq A_1,$$

where $v' = (v_r, v_z)$.

Moreover, Lemma 5.2 yields

$$(1.11) \quad \|v'\|_{V_2^1(\Omega_{\zeta_3}^{T_*})} \leq A_2.$$

Since $\{\zeta_1, \zeta_3\}$ is a partition of unity estimates (1.10) and (1.11) imply

$$(1.12) \quad \|v'\|_{V_2^1(\Omega^{T_*})} \leq A_1 + A_2.$$

The estimate as a priori is valid for any $T_* > 0$ (see Lemma 6.4 [Z.1] and Lemma 5.2). By Lemma 4.1 we have that

$$(1.13) \quad \|v_\varphi(t)\|_{H^1(\Omega)} \leq A_3, \quad t \in \mathbb{R}_+.$$

From (1.12) and (1.13) we derive

$$(1.14) \quad \|v(T_*)\|_{H^1(\Omega)} \leq A_1 + A_2 + A_3 \equiv \alpha.$$

Let T be such that

$$(1.15) \quad c_* T^{1/2} \alpha \leq 1,$$

and $T < T_*$ because $\|v(0)\|_{H^1(\Omega)} \leq \alpha$. Hence (1.14) takes the form

$$(1.16) \quad \|v(T)\|_{H^1(\Omega)} \leq \alpha.$$

Using that $\|v(T)\|_{H^1(\Omega)} \leq \alpha$ we can repeat the above considerations for any interval $[kT, (k+1)T]$, $k \in \mathbb{N}$.

The above extension in time of the local solution is possible because A_i , $i = 1, 2, 3$, do not depend on time and estimates (1.12), (1.13) are global a priori estimates (a priori means that it is supposed that there exists a solution to (1.1) such that $v \in W_2^{2,1}(\Omega \times \mathbb{R}_+)$).

2. Auxiliary results and notation

Definition 2.1. We introduce the spaces

$$V_2^0(\Omega^T) = \{u : \|u\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla u\|_{L_2(\Omega^T)} < \infty\}$$

and

$$\begin{aligned} V_2^0(\Omega \times (kT, (k+1)T)) \\ = \{u : \|u\|_{L_\infty(kT, (k+1)T; L_2(\Omega))} + \|\nabla u\|_{L_2(\Omega \times (kT, (k+1)T))} < \infty\} \end{aligned}$$

where $k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.

Lemma 2.2. Assume that $v_0 \in L_2(\Omega)$ and $d_1 > 0$ is a constant such that $\|v_0\|_{L_2(\Omega)} \leq d_1$. Then

$$(2.1) \quad \|v(t)\|_{L_2(\Omega)} \leq \|v_0\|_{L_2(\Omega)} \leq d_1.$$

Assume that $d_0 > 0$ is a constant such that $|\int_\Omega u_0 dx| \leq d_0$. Then

$$(2.2) \quad \left| \int_\Omega u(t) dx \right| = \left| \int_\Omega u_0 dx \right| \leq d_0.$$

Assume that $T > 0$ is fixed. Then

$$(2.3) \quad \|v\|_{V_2^0(\Omega \times (kT, t))}^2 \leq \frac{2}{\min\{1, \nu_2\}} \left(\frac{d_0^2}{\nu_1} e^{\nu_1 T} + \|v(kT)\|_{L_2(\Omega)}^2 \right),$$

where $t \in [kT, (k+1)T]$, $k \in \mathbb{N}_0$, $\nu_* = \nu_1 + \nu_2$, $\nu_i > 0$, $i = 1, 2$, $\nu_* = \frac{\nu}{c_k}$, c_k is the constant from the Korn inequality (2.5).

Proof. Multiplying (1.1)₁ by v , integrating over Ω , using (1.1)₂ and boundary conditions we obtain

$$(2.4) \quad \frac{d}{dt} \|v\|_{L_2(\Omega)}^2 + \nu E_\Omega(v) = 0,$$

where

$$E_{\Omega}(v) = \|\mathbb{D}(v)\|_{L_2(\Omega)}^2.$$

Integrating (2.4) with respect to time and omitting the second expression on the l.h.s. implies (2.1).

From [Z3, Ch. 4, Lemma 2.4] we have the following Korn inequality

$$(2.5) \quad \|v\|_{H^1(\Omega)}^2 \leq c_k \left(E_{\Omega}(v) + \left| \int_{\Omega} v \cdot \eta dx \right|^2 \right),$$

where $\eta = (-x_2, x_1, 0)$, $v \cdot \eta = rv_{\varphi} = u$.

Now we calculate the last term on the r.h.s. of (2.5). Multiplying (1.1)₁ by η , integrating over Ω , using (1.1)₂ we obtain

$$(2.6) \quad \frac{d}{dt} \int_{\Omega} v \cdot \eta dx - \int_{\Omega} v_i v_j \nabla_i \eta_j dx + \int_{\Omega} \mathbb{T}_{ij} \nabla_i \eta_j dx = 0,$$

where the summation convention over the repeated indices is assumed.

Using that $\nabla \eta$ is antisymmetric tensor we see that (2.6) implies

$$(2.7) \quad \frac{d}{dt} \int_{\Omega} v \cdot \eta dx = 0$$

Hence

$$(2.8) \quad \int_{\Omega} u(t) dx = \int_{\Omega} u_0 dx.$$

Employing (2.5) and (2.8) in (2.4) yields

$$(2.9) \quad \frac{d}{dt} \|v\|_{L_2(\Omega)}^2 + \nu_* \|v\|_{H^1(\Omega)}^2 \leq d_0^2,$$

where the inequality (2.2) was used and $\nu_* = \frac{\nu}{c_k}$.

Let $\nu_* = \nu_1 + \nu_2$, $\nu_i > 0$, $i = 1, 2$. Then (2.9) implies

$$(2.10) \quad \frac{d}{dt} (\|v\|_{L_2(\Omega)}^2 e^{\nu_1 t}) + \nu_2 \|v\|_{H^1(\Omega)}^2 e^{\nu_1 t} \leq d_0^2 e^{\nu_1 t}.$$

Integrating (2.10) with respect to time from kT to t yields

$$(2.11) \quad \begin{aligned} & \|v(t)\|_{L_2(\Omega)}^2 + \nu_2 e^{-\nu_1 t} \int_{kT}^t \|v(t')\|_{H^1(\Omega)}^2 e^{\nu_1 t'} dt' \\ & \leq \frac{d_0^2}{\nu_1} + e^{-\nu_1(t-kT)} \|v(kT)\|_{L_2(\Omega)}^2, \end{aligned}$$

where $t \in [kT, (k+1)T]$.

Omitting the first norm on the l.h.s. of (2.11) we obtain

$$(2.12) \quad \begin{aligned} \nu_2 \int_{kT}^t \|v(t')\|_{H^1(\Omega)}^2 dt' &\leq \frac{d_0^2}{\nu_1} e^{\nu_1(t-kT)} + \|v(kT)\|_{L_2(\Omega)}^2, \\ t &\in [kT, (k+1)T]. \end{aligned}$$

Hence, (2.11) and (2.12) imply (2.3). This concludes the proof.

From properties of $V_2^0(\Omega^T)$ we have

$$(2.13) \quad \|v\|_{L_q(0,T;L_p(\Omega))} \leq c_2 \|v\|_{V_2^0(\Omega^T)},$$

where

$$\frac{3}{p} + \frac{2}{q} \geq \frac{3}{2}.$$

Let us consider the problem

$$(2.14) \quad \begin{aligned} v_t - \operatorname{div} \mathbb{T}(v, p) &= f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} = 0, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2 && \text{on } S^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned}$$

From [ZZ] we have

Lemma 2.3. *Assume that $f \in L_s(\Omega^T)$, $v_0 \in W_s^{2-2/s}(\Omega)$, $s \in (1, \infty)$, $S_1 \subset C^2$. Then there exists a solution to problem (2.14) such that $v \in W_s^{2,1}(\Omega^T)$, $\nabla p \in L_s(\Omega^T)$ and there exists a constant $c = c(\Omega, s)$ such that*

$$(2.15) \quad \|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \leq c(\Omega, s) (\|f\|_{L_s(\Omega^T)} + \|v_0\|_{W_s^{2-2/s}(\Omega)}).$$

In view of Definition 1.1 equations (1.1)_{1,2} for the axially symmetric solutions assume the form

$$(2.16) \quad v_{r,t} + v \cdot \nabla v_r - \frac{v_\varphi^2}{r} - \nu \Delta v_r + \nu \frac{v_r}{r^2} = -p_{,r},$$

$$(2.17) \quad v_{\varphi,t} + v \cdot \nabla v_\varphi + \frac{v_r}{r} v_\varphi - \nu \Delta v_\varphi + \nu \frac{v_\varphi}{r^2} = 0$$

$$(2.18) \quad v_{z,t} + v \cdot \nabla v_z - \nu \Delta v_z = -p_{,z},$$

$$(2.19) \quad v_{r,r} + v_{z,z} = -\frac{v_r}{r},$$

where $v \cdot \nabla = v_r \partial_r + v_z \partial_z$, $\Delta u = \frac{1}{r}(ru_{,r})_{,r} + u_{,zz}$.

The slip-boundary conditions (1.1)_{3,4} on S_1 imply (see [Z3, Ch. 2])

$$(2.20) \quad v_r = 0, \quad v_{z,r} = 0, \quad v_{\varphi,r} = \frac{1}{R}v_\varphi \quad \text{on } S_1.$$

Lemma 2.4. *For the weak solutions to problem (1.1) we have the estimate*

$$(2.21) \quad \begin{aligned} & \|v\|_{V_2^0(\Omega \times (kT, t))}^2 + \left\| \frac{v_r}{r} \right\|_{L_2(\Omega \times (kT, t))}^2 + \left\| \frac{v_\varphi}{r} \right\|_{L_2(\Omega \times (kT, t))}^2 \\ & \leq c_0 \|v_0\|_{L_2(\Omega)}^2 \equiv d_1^2, \end{aligned}$$

where $c_0 = c(T+1)$, $t \in [kT, (k+1)T]$, $k \in \mathbb{N}_0$ and c is the constant from imbedding $H^1(\Omega) \subset L_2(S_1)$.

Proof. Multiplying (2.17) by v_φ , integrating over Ω and using boundary conditions (2.20) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} v_\varphi^2 dx + \int_{\Omega} \frac{v_r}{r} v_\varphi^2 dx + \nu \int_{\Omega} (v_{\varphi,r}^2 + v_{\varphi,z}^2) dx - \nu \int_{-a}^a v_\varphi^2 dz \\ & + \nu \int_{\Omega} \frac{v_\varphi^2}{r^2} dx = 0. \end{aligned}$$

Multiplying (2.16) by v_r , integrating over Ω and using boundary conditions (2.20) implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} v_r^2 dx - \int_{\Omega} \frac{v_\varphi^2}{r} v_r dx + \nu \int_{\Omega} (v_{r,r}^2 + v_{r,z}^2) dx + \nu \int_{\Omega} \frac{v_r^2}{r^2} dx \\ & = - \int_{\Omega} p_{,r} v_r dx. \end{aligned}$$

Multiplying (2.18) by v_z , integrating over Ω and using boundary conditions (2.20) we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} v_z^2 dx + \nu \int_{\Omega} (v_{z,r}^2 + v_{z,z}^2) dx = - \int_{\Omega} p_{,z} v_z dx.$$

Adding the above equations, using (2.19) and the inequality

$$\|v_\varphi\|_{L_2(S_1)}^2 \leq \varepsilon \|\nabla v_\varphi\|_{L_2(\Omega)}^2 + c(1/\varepsilon) \|v_\varphi\|_{L_2(\Omega)}^2$$

we obtain for sufficiently small ε ,

$$(2.22) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (v_r^2 + v_\varphi^2 + v_z^2) dx + \frac{\nu}{2} \int_{\Omega} (v_{r,r}^2 + v_{r,z}^2 + v_{\varphi,r}^2 + v_{\varphi,z}^2 + v_{z,r}^2 + v_{z,z}^2) dx \\ & + \nu \int_{\Omega} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx \leq c \int_{\Omega} v_\varphi^2 dx. \end{aligned}$$

Integrating (2.22) with respect to time from kT to $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$ and using (2.1) yields

$$\begin{aligned} & \|v(t)\|_{L_2(\Omega)}^2 + \nu \int_{kT}^t \int_{\Omega} (|v_{r,r}|^2 + |v_{r,z}|^2) dx dt' + \nu \int_{kT}^t \int_{\Omega} \left(\frac{v_r^2}{r^2} + \frac{v_z^2}{r^2} \right) dx dt' \\ & \leq cT d_1^2 + \|v(kT)\|_{L_2(\Omega)}^2. \end{aligned}$$

Applying again (2.1) gives (2.21). This ends the proof. \square

Let us introduce the component of vorticity

$$(2.23) \quad \chi = v_{r,z} - v_{z,r}.$$

Then χ is a solution to the problem (see [Z1, Z3])

$$(2.24) \quad \begin{aligned} & \chi_t + v \cdot \nabla \chi - \frac{v_r}{r} \chi - \nu \left[\left(r \left(\frac{\chi}{r} \right)_{,r} \right)_{,r} + \chi_{,zz} + 2 \left(\frac{\chi}{r} \right)_{,r} \right] \\ & = 2 \frac{v_\varphi v_{\varphi,z}}{r}, \\ & \chi|_{S_1} = 0, \quad \chi|_{S_2} \text{ satisfies periodic boundary conditions,} \\ & \chi|_{t=0} = \chi_0. \end{aligned}$$

Definition 2.5. By $V_2^k(\Omega^T)$, $k \in \mathbb{N} \cup \{0\}$, we denote the space of functions with the following finite norm

$$\|u\|_{V_2^k(\Omega^T)} = \text{esssup}_{t \in [0, T]} \|u\|_{H^k(\Omega)} + \|\nabla u\|_{L_2(0, T; H^k(\Omega))},$$

where

$$\|u\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D_x^\alpha u|^2 dx \right)^{1/2},$$

$$D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad \alpha = \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad i = 1, 2, 3.$$

Remark 2.6. From (2.21) we have that swirl $u = rv_\varphi$ satisfies

$$(2.25) \quad \left\| \frac{u}{r^2} \right\|_{L_2(\Omega^T)} \leq d_1,$$

but from [Z4] it follows that $u \in C^{\alpha, \alpha/2}(\Omega^T)$, $\alpha > 0$. Hence $u|_{r=0} = 0$.

Lemma 2.7. *Let d_1, d_2 be given positive constants. Let $\|v_0\|_{L_2(\Omega)} \leq d_1$, $\|u\|_{L_\infty(\Omega^T)} \leq d_2$ (see Assumptions 1, 2). Then*

$$(2.26) \quad \|v_\varphi\|_{L_4(\Omega^T)} \leq d_1^{1/2} d_2^{1/2}.$$

Proof follows from application of Lemma 2.4 and Lemma 2.1 from [Z4].

3. Local existence

We prove a local existence of solutions to problem (1.1) by the Leray-Schauder fixed point theorem applying the old idea of O. A. Ladyzhenskaya (see [L1, Ch. 4, Theorem 1']).

Let us consider the problem

$$(3.1) \quad \begin{aligned} v_t - \nu \Delta v + \nabla p &= -v \cdot \nabla v && \text{in } \Omega_\varepsilon^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega_\varepsilon^T, \\ \bar{n} \cdot v = 0, \quad \bar{n} \cdot \mathbb{T}(v, p) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2, && \text{on } S_1^T \cup S_\varepsilon^T, \\ \text{periodic boundary conditions} &&& \text{on } S_2^T, \\ v|_{t=0} &= v(0) && \text{in } \Omega_\varepsilon, \end{aligned}$$

where $\Omega_\varepsilon = \{x \in \Omega: r > \varepsilon\}$, $S_\varepsilon = \{x \in \mathbb{R}^3: r = \varepsilon, |z| < a\}$. Setting $\varepsilon = 0$ we obtain problem (1.1).

In view of [S, ZZ] there exists a constant c_0 such that solutions to (3.1) satisfy the inequality

$$(3.2) \quad \|v\|_{W_2^{2,1}(\Omega_\varepsilon^T)} \leq c_0(\|v \cdot \nabla v\|_{L_2(\Omega_\varepsilon^T)} + \|v(0)\|_{H^1(\Omega_\varepsilon)}).$$

Theorem 3.1. *Assume that $v(0) \in H^1(\Omega)$. Assume that T so small that*

$$(3.3) \quad c_* T^{1/2} \|v(0)\|_{H^1(\Omega)} \leq 1$$

where c_ is such constant that (3.10) is satisfied. Then there exists a solution to problem (1.1) such that $v \in W_2^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$, where T satisfies (3.3) and*

$$(3.4) \quad \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \leq 8c_0 \|v(0)\|_{H^1(\Omega)},$$

where c_0 is taken from (3.2).

Proof. Since we are going to prove the existence of solutions to problem (3.1) by the Leray-Schauder fixed point theorem we restrict the proof to show only estimate (3.4) because other steps of it are clear. We shall skip the index ε in Ω_ε for simplicity

Now we examine the first term on the r.h.s. of (3.2). We estimate it by

$$\begin{aligned} \left(\int_0^T dt \int_\Omega |v \cdot \nabla v|^2 dx \right)^{1/2} &\leq \left(\int_0^T \|v(t)\|_{L^\infty(\Omega)}^2 \|\nabla v(t)\|_{L_2(\Omega)}^2 dt \right)^{1/2} \\ &\leq \sup_t \|\nabla v(t)\|_{L_2(\Omega)} \left(\int_0^T \|v(t)\|_{L^\infty(\Omega)}^2 dt \right)^{1/2} \equiv I_1. \end{aligned}$$

From [BIN, Ch. 3, Sect. 15] we have

$$(3.5) \quad \|v\|_{L^\infty(\Omega)} \leq c_1 \|v_{xx}\|_{L_2(\Omega)}^{3/4} \|v\|_{L_2(\Omega)}^{1/4} + c_1 \|v\|_{L_2(\Omega)}.$$

Using (3.5) in I_1 yields

$$\begin{aligned} I_1 &\leq \sup_t \|\nabla v\|_{L_2(\Omega)} (c_1 T^{1/8} \|v_{xx}\|_{L_2(\Omega^T)}^{3/4} \sup_t \|v(t)\|_{L_2(\Omega)}^{1/4} \\ &\quad + c_1 T^{1/2} \sup_t \|v(t)\|_{L_2(\Omega)}) \equiv I_2. \end{aligned}$$

Employing

$$\sup_t \|\nabla v\|_{L_2(\Omega)} \leq \sup_t \|v(t)\|_{H^1(\Omega)} \leq c_2 (\|v\|_{W_2^{2,1}(\Omega^T)} + \|v(0)\|_{H^1(\Omega)})$$

and the energy estimate (2.1) in I_2 , we obtain from (3.2) the inequality

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq c_0 c_2 (\|v\|_{W_2^{2,1}(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}) \\ &\quad \cdot (c_1 T^{1/8} \|v_{xx}\|_{L_2(\Omega^T)}^{3/4} d_1^{1/4} + c_1 T^{1/2} d_1) + c_0 \|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

Assuming that T is so small that

$$(3.6) \quad c_0 c_2 c_1 (T^{1/8} \|v_{xx}\|_{L_2(\Omega^T)}^{3/4} d_1^{1/4} + T^{1/2} d_1) \leq 1/2$$

we obtain the inequality

$$\begin{aligned} (3.7) \quad \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq 2c_0 c_1 c_2 (T^{1/2} \|v\|_{W_2^{2,1}(\Omega^T)}^{3/4} d_1^{1/4} \\ &\quad + T^{1/2} d_1) \|v(0)\|_{H^1(\Omega)} + 2c_0 \|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

By the Young inequality applied to the first term on the r.h.s. of (3.7) we have

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq \frac{\varepsilon^{4/3}}{4/3} \|v\|_{W_2^{2,1}(\Omega^T)} \\ &+ \frac{1}{4\varepsilon^4} (2c_0c_1c_2T^{1/2}d_1^{1/4}\|v(0)\|_{H^1(\Omega)})^4 \\ &+ 2c_0(c_1c_2T^{1/2}d_1 + 1)\|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

Setting $\frac{\varepsilon^{4/3}}{4/3} = \frac{1}{2}$ so $\varepsilon = \left(\frac{2}{3}\right)^{3/4}$ the above inequality yields

$$\begin{aligned} \frac{1}{2}\|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq \frac{27}{32} (2c_0c_1c_2)^4 T^2 d_1 \|v(0)\|_{H^1(\Omega)}^4 \\ &+ 2c_0(c_1c_2T^{1/2}d_1 + 1)\|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

Simplifying the inequality yields

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq [27(c_0c_1c_2)^4 T^2 d_1 \|v(0)\|_{H^1(\Omega)}^3 \\ &+ 4c_0c_1c_2T^{1/2}d_1]\|v(0)\|_{H^1(\Omega)} + 4c_0\|v(0)\|_{H^1(\Omega)}. \end{aligned}$$

Assuming that T is so small that

$$(3.8) \quad 27(c_0c_1c_2)^4 T^2 d_1 \|v(0)\|_{H^1(\Omega)}^3 + 4c_0c_1c_2T^{1/2}d_1 \leq 4c_0$$

we obtain (3.4) after passing with ε to 0.

Using (3.4) in (3.6) we assume the stronger restriction

$$(3.9) \quad c_0c_1c_2[(4c_0)^{3/4}T^{1/8}d_1^{1/4} + T^{1/2}d_1] \leq 1/2.$$

Using that $d_1 \leq \|v(0)\|_{H^1(\Omega)}$ we find constants \bar{c}_i , $i = 1, \dots, 4$, such that (3.8) and (3.9) can be expressed in the form

$$\begin{aligned} (3.10) \quad \bar{c}_1 T^2 \|v(0)\|_{H^1(\Omega)}^4 + \bar{c}_2 T^{1/2} \|v(0)\|_{H^1(\Omega)} &\leq \frac{1}{2}, \\ \bar{c}_3 T^{1/8} \|v(0)\|_{H^1(\Omega)}^{1/4} + \bar{c}_4 T^{1/2} \|v(0)\|_{H^1(\Omega)} &\leq \frac{1}{2}. \end{aligned}$$

Therefore there exists a constant c_* satisfying

$$c_* T^{1/2} \|v(0)\|_{H^1(\Omega)} \leq 1$$

and such that (3.10) is satisfied. This concludes the proof.

4. Estimate for the angular component of velocity

In this section we prove the following estimate

$$(4.1) \quad \|v_\varphi(t)\|_{H^1(\Omega)} \leq d_5,$$

where d_5 does not depend on t .

For this purpose we consider the problem for v_φ

$$(4.2) \quad \begin{aligned} v_{\varphi,t} - \nu \Delta v_\varphi + v' \cdot \nabla v_\varphi + \frac{v_r}{r} v_\varphi + \nu \frac{v_\varphi}{r^2} &= 0 \quad \text{in } \Omega^T, \\ v_{\varphi,r} &= \frac{1}{r} v_\varphi \quad \text{on } S_1^T, \\ v_\varphi|_{S_2} &\text{ is periodic} \\ v_\varphi|_{t=0} &= v_\varphi(0) \quad \text{in } \Omega. \end{aligned}$$

Lemma 4.1. *Assume that there exists a local solution described by Theorem 3.1. Assume also that $A_0 = A_1 + A_2$ is finite (see assumptions from Section 1). Assume that*

$$\begin{aligned} d_5 &= \varphi(A_0)(d_1 + d_2) + c(\|v_\varphi(0)\|_{H^1(\Omega)} + \|v_\varphi(0)\|_{W_{5/2}^{1/5}(\Omega)}) \\ &\quad + \left\| \frac{v_\varphi(0)}{r} \right\|_{L_{\frac{27}{10}}(\Omega)} \leq A_3 \end{aligned}$$

is finite and does not depend on t . Then (4.1) holds for any $t \in \mathbb{R}_+$.

Proof. Multiplying (4.2)₁ by $v_{\varphi,t}$ and integrating the result over Ω yields

$$(4.3) \quad \begin{aligned} &\int_{\Omega} v_{\varphi,t}^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} |\nabla v_\varphi|^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} \frac{v_\varphi^2}{r^2} dx \\ &- \frac{\nu}{2} \frac{d}{dt} \int_{-a}^a v_\varphi^2|_{r=R} dz \leq \frac{\varepsilon_1}{2} \int_{\Omega} v_{\varphi,t}^2 dx + \frac{1}{2\varepsilon_1} \int_{\Omega} |v' \cdot \nabla v_\varphi|^2 dx \\ &+ \frac{\varepsilon_2}{2} \int_{\Omega} v_{\varphi,t}^2 dx + \frac{1}{2\varepsilon_2} \int_{\Omega} \left| \frac{v_r}{r} \right|^2 v_\varphi^2 dx. \end{aligned}$$

Setting $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ we get

$$(4.4) \quad \begin{aligned} &\frac{1}{2} \int_{\Omega} v_{\varphi,t}^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} |\nabla v_\varphi|^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} \frac{v_\varphi^2}{r^2} dx \\ &\leq \frac{\nu}{2} \frac{d}{dt} \int_{-a}^a v_\varphi^2|_{r=R} dz + \int_{\Omega} |v' \cdot \nabla v_\varphi|^2 dx + \int_{\Omega} \left| \frac{v_r}{r} \right|^2 v_\varphi^2 dx. \end{aligned}$$

Integrating (4.4) with respect to time and using estimates

$$(4.5) \quad \|v'\|_{L_{10}(\Omega^T)} \leq \varphi(A_0)$$

from [Z1, (6.19) and (6.20)],

$$(4.6) \quad \left\| \frac{v_r}{r} \right\|_{L_{10}(\Omega^T)} \leq \varphi(A_0)$$

from [Z1, (6.38)], and (see (4.17))

$$(4.7) \quad \|v_\varphi|_{r=R}\|_{L_\infty(\Omega^T)} \leq c\|u_0\|_{L_\infty(\Omega)}$$

we obtain

$$(4.8) \quad \begin{aligned} & \int_{\Omega^t} v_{\varphi,t}^2 dx dt + \nu \int_{\Omega} |\nabla v_\varphi(t)|^2 dx + \nu \int_{\Omega} \frac{v_\varphi^2(t)}{r^2} dx \\ & \leq \nu \int_{-a}^a v_\varphi^2|_{r=R} dz + \nu \int_{\Omega} |\nabla v_\varphi(0)|^2 dx + \nu \int_{\Omega} \frac{v_\varphi^2(0)}{r^2} dx \\ & \quad + \varphi(A_0)(\|\nabla v_\varphi\|_{L_{5/2}(\Omega^t)}^2 + \|v_\varphi\|_{L_{5/2}(\Omega^t)}^2), \end{aligned}$$

where the first integral on the r.h.s. is bounded by

$$c\|u(0)\|_{L_\infty(\Omega)}.$$

To estimate the norms from the last term on the r.h.s. of (4.8) we introduce the Green function to problem (4.2). Let us denote it by G . Then we can express (4.2) in the integral form

$$(4.9) \quad \begin{aligned} v_\varphi(x, t) = & \int_{\Omega^t} \nabla_y G(x - y, t - \tau) v' v_\varphi dy d\tau \\ & - \int_{\Omega^t} G(x - y, t - \tau) \left(\frac{v_r}{r} v_\varphi + \nu \frac{v_\varphi}{r^2} \right) dy d\tau \\ & + \int_{\Omega} G(x - y, t) v_\varphi(y, 0) dy + \int_{S_1^t} G(x - z, t - \tau) v_\varphi(R, z) dz d\tau. \end{aligned}$$

From (4.9) we have

$$(4.10) \quad \begin{aligned} \|v_\varphi\|_{W_\sigma^{1,1/2}(\Omega^T)} \leq & c \left(\|v' v_\varphi\|_{L_\sigma(\Omega^T)} + \left\| \frac{v_r}{r} v_\varphi \right\|_{L_{\frac{5\sigma}{5+\sigma}}(\Omega^T)} \right. \\ & \left. + \left\| \frac{v_\varphi}{r^2} \right\|_{L_{\frac{5\sigma}{5+\sigma}}(\Omega^T)} + \|v_\varphi(0)\|_{W_\sigma^{1-\frac{2}{\sigma}}(\Omega)} + c\|u(0)\|_{L_\infty(\Omega)} \right), \end{aligned}$$

where the last norm estimates the last integral on the r.h.s. of (4.9).

To estimate the norm $\|\nabla v_\varphi\|_{L_{5/2}(\Omega^T)}$ appearing on the r.h.s. of (4.8) we assume $\sigma = \frac{5}{2}$.

Now we estimate the particular terms on the r.h.s. of (4.10). We bound the first term by

$$\begin{aligned} \left\| \frac{v'}{r} r v_\varphi \right\|_{L_{5/2}(\Omega^T)} &= \left\| \frac{v'}{r} u \right\|_{L_{5/2}(\Omega^T)} \leq \|u(0)\|_{L_\infty(\Omega)} \left\| \frac{v'}{r} \right\|_{L_{5/2}(\Omega^T)} \\ &\leq \varphi(A_0) \|u(0)\|_{L_\infty(\Omega)}, \end{aligned}$$

where we used (6.19), (6.20), (6.38) from [Z1] (see (4.6), (4.7)).

The second integral on the r.h.s. of (4.10) is estimated by

$$\left\| v_r \frac{v_\varphi}{r} \right\|_{L_{\frac{5}{3}}(\Omega^T)} \leq \|v_r\|_{L_{10}(\Omega^T)} \left\| \frac{v_\varphi}{r} \right\|_{L_2(\Omega^T)} \leq \varphi(A_0) d_1$$

where we used (6.19) and (6.20) from [Z1].

Assuming that $v_\varphi \geq 1$, because otherwise we have regularity of axially symmetric solutions (see [NP1, NP2, P]), the third integral on the r.h.s. of (4.10) is bounded by

$$(4.11) \quad \left\| \frac{v_\varphi}{r} \right\|_{L_{\frac{10}{3}}(\Omega^T)}.$$

Finally, the fourth term on the r.h.s. of (4.10) equals

$$\|v_\varphi(0)\|_{W_{\frac{5}{2}}^{1-4/5}(\Omega)}$$

To estimate (4.11) we introduce the quantity $\omega = \frac{v_\varphi}{r}$ and multiply (4.2)₁ by $\frac{1}{r}$. Then we obtain the following equation for ω

$$\begin{aligned} (4.12) \quad & \omega_{,t} + v' \cdot \nabla \omega + \frac{2v_r}{r} \omega - \nu \Delta \omega - \frac{2\nu}{r} \omega_{,r} = 0 \quad \text{in } \Omega^T, \\ & \omega_{,r} = 0 \quad \text{on } S_1^T, \\ & \omega|_{S_2} - \text{periodic with respect to } z, \\ & \omega|_{t=0} = \omega(0) \quad \text{in } \Omega. \end{aligned}$$

Multiplying (4.12) by $\omega|\omega|^{s-2}$ and integrating over Ω yields

$$\begin{aligned} (4.13) \quad & \frac{1}{s} \frac{d}{dt} \int_{\Omega} |\omega|^s dx + \frac{4\nu(s-1)}{s^2} \int_{\Omega} |\nabla |\omega|^{s/2}|^2 dx + \frac{2\nu}{s} \int_{-a}^a |\omega|^s|_{r=0} dz \\ & = -2 \int_{\Omega} \frac{v_r}{r} |\omega|^s dx + \frac{2\nu}{s} \int_{-a}^a |\omega|_{r=R}^s dz. \end{aligned}$$

Applying the trace estimate

$$\int_{-a}^a |\omega^{s/2}|^2 dz \leq \varepsilon \int_{\Omega} |\nabla |\omega|^{s/2}|^2 dx + \frac{c}{\varepsilon} \int_{\Omega} |\omega|^s dx$$

to the last term on the r.h.s. of (4.13) and assuming that $\varepsilon = \frac{s-1}{s}$ we obtain

$$(4.14) \quad \begin{aligned} & \frac{1}{s} \frac{d}{dt} \int_{\Omega} |\omega|^s dx + \frac{2\nu(s-1)}{s^2} \int_{\Omega} |\nabla |\omega|^{s/2}|^2 dx \\ & \leq -2 \int_{\Omega} \frac{v_r}{r} |\omega|^s dx + \frac{cs}{s-1} \int_{\Omega} |\omega|^s dx. \end{aligned}$$

Integrating (4.14) with respect to time we get

$$(4.15) \quad \begin{aligned} & \frac{1}{s} \int_{\Omega} |\omega|^s dx + \frac{2\nu(s-1)}{s^2} \int_{\Omega^t} |\nabla |\omega|^{s/2}|^2 dx dt' \\ & \leq 2 \int_{\Omega^t} \left| \frac{v_r}{r} \right| |\omega|^s dx dt' + \frac{2\nu}{s} \int_{\Omega^t} |\omega|^s dx dt' + \frac{1}{s} \int_{\Omega} |\omega(0)|^s dx, \end{aligned}$$

where the first integral on the r.h.s. is bounded by

$$2 \left\| \frac{v_r}{r} \right\|_{L_{10}(\Omega^t)} \|\omega\|_{L_{\frac{10}{9}s}(\Omega^t)}^s \leq \varphi(A_0) \|\omega\|_{L_{\frac{10}{9}s}(\Omega^t)}^s,$$

where (6.38) from [Z1] was used.

Setting $s = \frac{9}{5}$ and using the energy estimate (2.21) we derive

$$(4.16) \quad \|\omega\|_{L_3(\Omega^t)} \leq \varphi(A_0, d_1) d_1 + c \|\omega(0)\|_{L_{\frac{9}{5}}(\Omega)},$$

where φ and c do not depend on t .

Next, setting $s = \frac{27}{10}$ in (4.15) and employing (4.16) we obtain from (4.15) the estimate

$$(4.17) \quad \|\omega\|_{L_{9/2}(\Omega^t)} \leq \varphi(A_0, d_1) d_1 + c \|\omega(0)\|_{L_{\frac{27}{10}}(\Omega)},$$

where φ and c do not depend on t .

The inequality is sufficient to estimate (4.11). Summarizing, we have

$$(4.18) \quad \begin{aligned} & \|v_{\varphi}\|_{L_{5/2}(\Omega^T)} + \|\nabla v_{\varphi}\|_{L_{5/2}(\Omega^T)} \leq \varphi(A_0, d_1) [\|u(0)\|_{L_{\infty}(\Omega)} + d_1] \\ & + c \left(\|v_{\varphi}(0)\|_{W_{5/2}^{1/5}(\Omega)} + \left\| \frac{v_{\varphi}(0)}{r} \right\|_{L_{\frac{27}{10}}(\Omega)} + \|u(0)\|_{L_{\infty}(\Omega)} \right). \end{aligned}$$

Using (4.18) in (4.8), applying the trace estimate to the first integral on the r.h.s. of (4.8) and employing (2.21) we obtain

$$(4.19) \quad \begin{aligned} \|v_\varphi(t)\|_{H^1(\Omega)} &\leq \varphi(A_0, d_1)(\|u(0)\|_{L^\infty(\Omega)} + d_1) \\ &+ c \left(\|v_\varphi(0)\|_{H^1(\Omega)} + \|v_\varphi(0)\|_{W^{1/5}_{5/2}(\Omega)} + \left\| \frac{v_\varphi(0)}{r} \right\|_{L_{27/10}(\Omega)} \right. \\ &\quad \left. + \|u(0)\|_{L^\infty(\Omega)} \right). \end{aligned}$$

This concludes the proof.

5. Existence

Let $\zeta_1 = \zeta_1(r)$ be a smooth function such that $\zeta_1(r) = 0$ for $r \geq 2r_0$ and $\zeta_1(r) = 1$ for $r \leq r_0$. By $\zeta_2 = \zeta_2(r)$ we denote such smooth function that $\zeta_2(r) = 1$ for $r \geq r_0$ and $\zeta_2(r) = 0$ for $r \leq r_0/2$. Let us introduce the notation:

$$v^{(i)} = v' \zeta_i, \quad \chi^{(i)} = \chi \zeta_i^2, \quad v_\varphi^{(k)} = v_\varphi \zeta_i, \quad i = 1, 2.$$

From Lemma 6.2 in [Z1] we have

Lemma 5.1. *Assume that v is a weak solution to problem (1.1). Assume that $v \in W^{2,1}_2(\Omega^T)$ and $u \in C^{1/2,1/4}(\Omega^T)$. Assume that $\frac{[v_\varphi^{(1)}(0)]^2}{r}, \frac{\chi^{(1)}(0)}{r} \in L_2(\Omega)$. Then*

$$(5.1) \quad \begin{aligned} \|v^{(1)}\|_{V^1_2(\Omega^T)} &\leq \varphi(d_1, d_2, 1/r_0) \left[1 + \left\| \frac{[v_\varphi^{(1)}(0)]^2}{r} \right\|_{L_2(\Omega)} \right. \\ &\quad \left. + \left\| \frac{\chi^{(1)}(0)}{r} \right\|_{L_2(\Omega)} \right]. \end{aligned}$$

In view of (2.24) $\chi^{(2)}$ is a solution to the problem

$$(5.2) \quad \begin{aligned} &\chi_{,t}^{(2)} + v \cdot \nabla \chi^{(2)} - \frac{v_r}{r} \chi^{(2)} - v \cdot \nabla \zeta_2 \chi \\ &\quad - \nu \left[\left(r \left(\frac{\chi^{(2)}}{r} \right)_{,r} \right)_{,r} + \chi_{,zz}^{(2)} + 2 \left(\frac{\chi^{(2)}}{r} \right)_{,r} \right] \\ &\quad + \nu \left[\frac{\chi}{r} \zeta_{2,r}^2 + 2 \left(\frac{\chi}{r} \right)_{,r} \zeta_{2,r}^2 - \chi \zeta_{2,rr}^2 \right] = \frac{2v_\varphi^{(2)} v_{\varphi,z}^{(2)}}{r}, \\ &\chi^{(2)}|_{S_1} = 0, \quad \chi^{(2)}|_{t=0} = \chi^{(2)}(0), \\ &\chi^{(2)}|_{S_2} \text{ satisfic periodic boundary conditions with respect to } z. \end{aligned}$$

Lemma 5.2. Assume that v is a weak solution to problem (1.1). Assume that $v_\varphi(0) \in L_{49/18}(\Omega_{\zeta_2})$, $\chi^{(2)}(0) \in L_2(\Omega)$. Then

$$(5.3) \quad \|\chi\|_{V_2^0(\Omega_{\bar{\zeta}_2}^T)} \leq c(1/r_0, d_1, d_2)(d_1^2 + \|\chi^{(2)}(0)\|_{L_2(\Omega)}),$$

where $\Omega_{\bar{\zeta}_2} = \{x \in \Omega : \zeta_2(r) = 1\}$.

Proof. Multiplying (5.2) by $\frac{\chi^{(2)}}{r^2}$, integrating over Ω and using the boundary conditions yields

$$(5.4) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \frac{\chi^{(2)}}{r} \right|^2 dx + \nu \int_{\Omega} \left| \nabla \frac{\chi^{(2)}}{r} \right|^2 dx = \int_{\Omega} v \cdot \nabla \zeta_2 \chi \frac{\chi^{(2)}}{r^2} dx \\ & - \nu \int_{\Omega} \left[\frac{\chi}{r} \zeta_{2,r}^2 + 2 \left(\frac{\chi}{r} \right)_{,r} \zeta_{2,r}^2 - \chi \zeta_{2,rr}^2 \right] \frac{\chi^{(2)}}{r^2} dx \\ & + 2 \int_{\Omega} \frac{v_\varphi^{(2)} v_{\varphi,z}^{(2)}}{r} \frac{\chi^{(2)}}{r^2} dx \end{aligned}$$

The first term on the r.h.s. is estimated by

$$\varepsilon \left\| \frac{\chi^{(2)}}{r} \right\|_{L_6(\Omega)}^2 + c(1/\varepsilon, 1/r_0) \|v_r\|_{L_2(\Omega)}^2 \left\| \frac{\chi}{r} \right\|_{L_3(\Omega_{\zeta_2,r})}^2,$$

where $\Omega_{\zeta_2,r} = \Omega \cap \text{supp } \zeta_{2,r}$

The second term on the r.h.s. of (5.4) we express in the form

$$-\nu \int_{\Omega} \left(\frac{\chi^2}{r^3} \zeta_{2,r}^2 - \frac{\chi^2}{r^2} \zeta_{2,rr}^2 \right) \zeta_2^2 dx - 2\nu \int_{\Omega} \left(\frac{\chi}{r} \right)_{,r} \frac{\chi}{r} \frac{1}{r} \zeta_{2,r}^2 \zeta_2^2 dx \equiv I.$$

Integrating by parts the second integral in I takes the form

$$\nu \int_{\Omega} \frac{\chi^2}{r^2} (\zeta_2^2 \zeta_{2,r}^2)_{,r} dr dz.$$

Hence,

$$|I| \leq c(1/r_0) \int_{\Omega} \chi^2 dx.$$

Finally, the last integral on the r.h.s. of (5.4) is estimated by

$$\varepsilon \int_{\Omega} \left| \nabla \frac{\chi^{(2)}}{r} \right|^2 dx + c(1/\varepsilon) \int_{\Omega} \frac{|v_\varphi^{(2)}|^4}{r^4} dx.$$

Using the above estimates in (5.4), assuming that ε is sufficiently small and integrating the result with respect to time we obtain

$$(5.5) \quad \begin{aligned} \left\| \frac{\chi^{(2)}}{r} \right\|_{V_2^0(\Omega^t)}^2 &\leq c(1/r_0)d_1^2 \left\| \frac{\chi}{r} \right\|_{L_2(0,t;L_3(\Omega_{\zeta_2,r}))}^2 \\ &+ c(1/r_0)d_1^2 + c(1/r_0) \int_{\Omega^t} |v_\varphi^{(2)}|^4 dxdt + \left\| \frac{\chi^{(2)}(0)}{r} \right\|_{L_2(\Omega)}^2, \end{aligned}$$

where estimate (2.21) is employed.

In view of (2.26) and the interpolation

$$\begin{aligned} \left\| \frac{\chi}{r} \right\|_{L_2(0,T;L_3(\Omega_{\zeta_2,r}))}^2 &\leq \varepsilon \left\| \nabla \frac{\chi}{r} \right\|_{L_2(\Omega_{\zeta_2}^t)}^2 + c(1/\varepsilon) \left\| \frac{\chi}{r} \right\|_{L_2(\Omega_{\zeta_2}^t)}^2 \\ &\leq \varepsilon \left\| \nabla \frac{\chi}{r} \right\|_{L_2(\Omega_{\zeta_2}^t)}^2 + c(1/\varepsilon, 1/r_0)d_1^2, (1 + d_2^2), \end{aligned}$$

we obtain from (5.5) the inequality

$$(5.6) \quad \left\| \frac{\chi}{r} \right\|_{V_2^0(\Omega_{\zeta_2}^t)}^2 \leq \varepsilon \left\| \frac{\chi}{r} \right\|_{V_2^0(\Omega_{\zeta_2}^t)}^2 + c(1/\varepsilon, 1/r_0, d_1, d_2)d_1^2 + \left\| \frac{\chi^{(2)}(0)}{r} \right\|_{L_2(\Omega)}^2.$$

By the local iteration argument (see [LSU, Ch. 4, Sect. 10]) inequality (5.6) for $\varepsilon < 1$ implies (5.3). This concludes the proof.

In view of (5.3) we have the estimate for $\chi \in V_2^0(\Omega_{\zeta_2}^T)$, where $\Omega_{\zeta_2} = \{x \in \Omega : \zeta_2(r) = 1\}$. Take a function $\zeta_3 = \zeta_3(r)$ such that $\text{supp } \zeta_3 \subset \Omega_{\zeta_2}$. Now we consider the problem

$$(5.7) \quad \begin{aligned} v_{r,z} - v_{z,r} &= \chi, \\ v_{r,r} + v_{z,z} + \frac{v_r}{r} &= 0, \end{aligned}$$

in Ω_{ζ_2} . Hence, to formulate the problem more precisely we multiply (5.7) by ζ_3 and introduce the notation

$$v^{(3)} = v\zeta_3, \quad \chi^{(3)} = \chi\zeta_3.$$

Then (5.7) takes the form

$$(5.8) \quad \begin{aligned} v_{r,z}^{(3)} - v_{z,r}^{(3)} &= \chi^{(3)} - v_z \zeta_{3,r} && \text{in } \Omega_{\zeta_2}, \\ v_{r,r}^{(3)} + v_{z,z}^{(3)} &= -\frac{v^{(3)}}{r} + v_r \zeta_{3,r} && \text{in } \Omega_{\zeta_2}, \\ v^{(3)} \cdot \bar{n}|_{S_1} &= 0, \quad v^{(3)}|_{S_2} && \text{satisfies periodic conditions,} \\ v^{(3)}|_{r=r_0} &= 0. \end{aligned}$$

Since the r.h.s. of (5.8) belongs to $V_2^0(\Omega_{\bar{\zeta}_2}^T)$ we obtain the estimate

$$(5.9) \quad \|v'^{(3)}\|_{V_2^1(\Omega_{\bar{\zeta}_2}^T)} \leq c(\|\chi^{(3)}\|_{V_2^0(\Omega_{\bar{\zeta}_2}^T)} + d_1).$$

Hence, from (5.1) and (5.9) we obtain the estimate

$$(5.10) \quad \begin{aligned} \|v'\|_{V_2^1(\Omega^T)} &\leq \varphi(d_1, d_2, 1/r_0) \left[d_1^2 + \left\| \frac{|v_\varphi^{(1)}(0)|^2}{r} \right\|_{L_2(\Omega)} \right. \\ &\quad \left. + \left\| \frac{\chi^{(1)}(0)}{r} \right\|_{L_2(\Omega)} + \|\chi^{(2)}(0)\|_{L_2(\Omega)} \right] \equiv d_3, \end{aligned}$$

where $v' = (v_r, v_z)$.

From [Z2, proof of Lemma 3.7] and (5.10) we have

$$(5.11) \quad \|v'\|_{L_{10}(\Omega^T)} + \|\nabla v'\|_{L_{10/3}(\Omega^T)} \leq cd_3 \equiv A_0.$$

Theorem 5.3. (*Global existence*) (*Proof of Theorem A*) *Let the Assumptions from Section 1 hold. Then there exists a solution to problem (1.1) such that $v \in W_2^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$, $T \in R_+$ and*

$$(5.12) \quad \|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \leq A_1 + A_2 + A_3 \equiv \alpha,$$

where $t \in [(k-1)T_0, kT_0]$, $k \in \mathbb{N}$, T_0 is such that

$$(5.13) \quad c_* T_0^{1/2} \alpha \leq 1$$

and c_* is introduced in (3.3).

Proof. Assume that $v(0) \in H^1(\Omega)$ is given. Then Theorem 3.1 implies existence of local solution $v \in W_2^{2,1}(\Omega^{T_*})$, where T_* satisfies (3.3). By imbedding $v \in L_{10}(\Omega^{T_*})$ so [Z4] yields

$$(5.14) \quad \|u\|_{C(0, T_*; C^{1/2}(\Omega))} \leq d_4,$$

where d_4 does not depend on the local solution. Then (5.10) implies that

$$(5.15) \quad \|v'(T_*)\|_{H^1(\Omega)} \leq d_3 \leq A_1 + A_2 \equiv A_0,$$

where A_0 does not depend on T_* .

Hence (5.15) and (4.1) imply

$$(5.16) \quad \|v(T_*)\|_{H^1(\Omega)} \leq d_3 + d_5 \leq A_1 + A_2 + A_3 \equiv \alpha.$$

Choosing $T_0 < T_*$ satisfying (5.13) we can continue the above considerations in the interval $[T_0, 2T_0]$. In this way we show that

$$(5.17) \quad \|v(kT_0)\|_{H^1(\Omega)} \leq \alpha.$$

This concludes the proof.

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